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The Estimation of the Grant Element of Loans Reconsidered

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This paper examines critically the formulae which are frequently used in the calculations of the grant element of loans. Given the formula derived by Beenhakker (1976), which has been expanded into a more general form, the grant element of foreign assistance received by the Sudan during the period 1958–1979 is calculated. The grant element was found to be low, reflecting hard terms of borrowing.

1. INTRODUCTION

It has been widely accepted that when loans are made on concessionary terms, they contain an aid component, or a grant element, which can be estimated in cash terms and regarded as a cost (to donors) or a benefit (to recipients) associated with such loans. The grant element is thus defined as the difference between the nominal value of the loan and the present value of all future repayments (amortization and interest) discounted by a proper discount rate.

The grant element method has the advantage of expressing the nature of loans (whether soft or hard) across donor sources, or of a whole loan programme, in terms of a single parameter. Thus, it facilitates the ranking of donors by their aid programmes and helps in distinguishing the desirable form(s) of credits as well as the corresponding sources.

Given the terms of borrowing, the grant element or the aid component of loans can be estimated by applying any appropriate formula. The impetus of the most commonly used formula goes back to Ohlin (1966). However, the application of this formula is limited to certain types of loans and hence it cannot be generalized. Therefore, the purpose of this paper is to examine critically Ohlin's formula and determine an alternative formulation which lends itself to a wider range of applicability by focusing on the less limiting formula of Beenhakker (1976).

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The next section of this paper discusses the factors which determine the grant element of loans, while Section 3 examines critically the formulae by which the grant element can be calculated, and it determines the formula which has been applied in Section 4 to the case of the Sudan during the period 1958–1979. The final section offers some concluding remarks.

2. THE DETERMINANTS OF THE GRANT ELEMENT

The factors which determine the value of the grant element are mainly three: the rate of interest attached to the loan which is the major one, the grace period which lies between the date of disbursement until the repayments start (usually during this period only the interest is paid), and the maturity period by the end of which the repayments obligations terminate. These factors can be incorporated into this formula:

$$A = \left[\frac{F - \sum_{n=1}^{N} \frac{P_n}{(1+i)^n}}{F} \right] 100 \qquad \ldots \qquad \ldots \qquad (1)$$

or

$$A = \left(\frac{F - PV}{F} \right) 100 \qquad \ldots \qquad \ldots \qquad (2)$$

where A is the grant element as a percentage of the face-value of the loan, F is the face-value of the loan, P_n is the total payment of principal and interest in year n, N is the maturity period, i is the discount rate, and PV is the present value of future repayments on the loan.

The lower the rate of interest and the higher the grace and maturity periods, the higher will be the grant element. Given these factors, the grant element can be calculated for different combinations of them in order to determine, for instance, by how many years a one percent increase in the interest rate can be offset by a corresponding increase in the grace and/or maturity periods.

3. THE ESTIMATION OF THE GRANT ELEMENT

Given the terms of borrowing, several formulations have been suggested for accomplishing the calculations of the grant element. As stated earlier, the most commonly used formula is attributable to Ohlin (1966). Assuming a constant stream of debt servicing payments (according to which the debtor will surrender a constant annual payment of the principal and interest when the grace period elapses), this form applies for long-term loans:[1]

[1] Debt-servicing payments can also be made at an increasing rate (i.e., in each successive period the payable instalment increases) or at a decreasing rate over time.

$$A = \left(1 - \frac{r}{i}\right)\left(1 - \frac{e^{-iG} - e^{-iN}}{i(N-G)}\right) \quad \ldots \quad \ldots \quad (3)$$

where r is the interest rate attached to the loan, G is the grace period, and i and N are as defined before.

Ohlin's formula, like most of the conventional formulae, can be criticized for being very simplistic, to the extent that the accuracy of the estimated values of the grant element becomes questionable. In addition, formula (3) operates only when the discount rate is different from the interest rate attached to the loan (i.e., when $i \neq r$). If $i = r$, any loan, irrespective of its length of maturity and grace period, will yield a zero grant element. But it is clear from Equation (2) that the grant element will be positive if $PV < F$, equal to zero if $PV = F$, and negative if $PV > F$, i.e., the interest rate is a necessary but not a sufficient determinant of the grant element. The effect of the maturity and grace periods is also important as the length of these periods may counteract any increase in the interest rate attached to the loan and hence maintain the value of the grant element all the same.[2]

Furthermore, when the grace period is equivalent to the maturity period (i.e., a bullet loan), formula (3) reduces to:

$$A = (1 - r/i)\ (1 - e^{-iN}) \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

This formula, which is arrived at by applying L'Hospital's Rule to formula (3) when $G = N$, unnecessarily overestimates the value of the grant element as will be shown in the coming discussion.[3]

Moreover, Ohlin maintains that when the rate of discount is too high, it is possible to use the following 'rules of thumb': "each concession of one percentage point in the interest rate gives rise to a grant element of 4 percent of the face-value for a 10-year loan; 7 percent for a 20-year loan; 9 percent for a 30-year loan, and 10 percent for a 40-year loan" [Ohlin (1966), p. 103].

A more general formula which deals with the different forms of loans, and also

[2] It is implicit that Ohlin's formulation distinguishes between short-term and long-term loans. Hence, for short-term loans, this approximation is suggested:

$$A = \frac{(i - r)N}{2}$$

where all the terms are as defined before. In fact, this approximation is valid when $iN < 1$ and $G = 0$, which is a specific definition of short-term loans.

[3] Those who are interested in the derivation of formula (4) can pursue the exercise themselves. However, it should be mentioned that a similar application of L'Hospital's Rule is contained in Appendix (2).

operates even when the rate of interest attached to the loan is equal to the discount rate (i.e., the effect of the grace and maturity periods is accounted for), has been derived by Beenhakker (1976). It is clear from Equation (2) that the grant element can be calculated by determining first the present value of all future repayments (*PV*). This can be done by using the "Zeta" transformation for discrete time-series analysis (see Appendix 1), and the resulting formula would be:

$$PV = rF \left[1-(1+i)^{-G} + (1+r)^{N-G+1} \times \left((1+i)^{-G} - (1+i)^{-N}\right) \div \left((1+r)^{N-G+1} - (1+r)\right) \right] / i \quad \ldots \quad \ldots \quad (5)$$

where all the variables are as defined before.

However, formula (5), like Ohlin's formulation, does not also deal with the case of a bullet loan, i.e., when $G = N$. Therefore, L'Hospital's Rule is also applied to formula (5) and the resulting form [as derived in Appendix (2)] that can be used to determine the present value of debt servicing on loans when $G = N$ is:

$$PV = rF \left[1-(1+i)^{-G} + (1+r) \times \left((1+i)^{-G} \log.(1+i)\right) \div \left((1+r) \log (1+r)\right) \right] / i \quad \ldots \quad \ldots \quad (6)$$

It can be noticed that formulae (4) and (6) are derived by applying L'Hospital's Rule to formulae (3) and (5), when $G = N$, respectively. As pointed out earlier, when $G = N$, Ohlin's formula [i.e., formula (4)] would overestimate the values of the grant element. In Table 1, the values of the grant element obtained through the application of formulae (4) and (6) are compared. It is clear that formula (4) overestimates the values of the grant element.

4. THE GRANT ELEMENT OF FOREIGN ASSISTANCE TO THE SUDAN 1958–1979

The grant element of loans to the Sudan is calculated from the recipient's point of view (in order to assess the embodied benefit or the concessional element in loans) by applying formula (5) to Sudanese data. The data are based on contracted official loans because exact figures on the terms of borrowing and conditions were not available on the basis of actual flow of funds.

The three different forms of official foreign assistance contracted by the Sudan during the period 1958–1979 are compiled in Table 2. The Sudanese currency is used as a unit of measurement in order to account for the effect of the various exchange rates, i.e., all the foreign loans were converted into domestic Sudanese pounds, according to the prevailing exchange rates when these loans were contracted.

Table 1

A Comparison of the Grant Element when the Maturity and the Grace Periods are Equal

Terms of Borrowing			Grant Element Using	
Rate of Interest	Maturity Period	Grace Period	Formula (4)	Formula (6)
(Percent)	Years		At 10 Percent Discount Rate	
0.75	15	15	71.9	59.3
1.0	10	10	56.9	40.6
3.0	8	8	38.5	31.5
3.5	5	5	25.6	17.7
5.0	4	4	16.5	13.2

Source: Own estimates based on data compiled from various issues of the Annual Report of the Bank of Sudan and the records of the Sudanese Ministries of National Planning, and Finance and National Economy.

Table 2

Sources and Size of Official Loans Contracted by the Sudan during the Period 1958–1979

Source of Loans	Total Amount (L.S. Million)	Percent Share in Total Borrowing
(1) Bilateral Agreements	802.76620	66.7
(a) USA and West European Countries	245.73118	20.4
(b) Arab Countries	432.42502	35.9
(c) East European Countries	124.61000	10.4
(2) Multilateral Agreements	223.71700	18.6
(3) Borrowing from Private Sources	176.53000	14.7
Total	1203.01320	100.0

Source: Based on data from the same source as Table 1.

It can be seen from Table 2 that the share of bilateral assistance in the total flows was the highest, followed by the share of multilateral aid, and then borrowing from private sources. The amount of loans contracted with the Arab countries was the largest, and it constituted 35.9 percent of the total flows.

On the other hand, Table 3 reveals that the grant element of contracted Sudanese loans fluctuated greatly over time, and so did the terms of borrowing.[4] The overall averages of the grant element tend to be 31 percent and 39 percent at the discount rates of 8 percent and 10 percent, respectively.[5] Regarding the terms of borrowing, the average rate of interest, the length of maturity, and the grace period were 4 percent, 16 and 6 years, respectively.

5. CONCLUDING REMARKS

This paper has examined critically the conventional formulae which are frequently used in the estimates of the grant element of loans. Particular attention is given to the commonly used formula of Ohlin (1966) which is constrained by several limitations. Consequently, a more practical formula, which was originally developed by Beenhakker (1976) and expanded into a more general form, has been applied to the foreign capital inflows received by the Sudan during the period 1958–1979.

The results show that the grant element was low, averaging 31 percent and 39 percent at the discount rates of 8 percent and 10 percent respectively. On the other hand, the terms of borrowing averaged 4 percent, 16 and 6 years, regarding the interest rate attached to the loans, the length of maturity, and the grace period respectively.

[4]The grant element is calculated by using annual weighted shares of individual loans in the total inflows, i.e., by multiplying the grant element of each loan (A) received in a specific year by the nominal value of the loan (F) and dividing the product by the total annual amount of loans received during that year. This relationship can be expressed as:

$$(Am \times Fm) \,/\, \sum_{m=1}^{M} Fm$$

where $m = 1 \ldots . M$, is the number of loans received in a given year.

[5]The 8 percent discount rate is assumed to represent the world market rate of interest being proxied by the average annual rate of the UK money-markets and the Euro Dollar market during the period 1958–1979. This rate acts as the rate of interest at which the Sudan might have had to borrow in the absence of aid, and it has been calculated from the IMF International Financial Statistics; whilst the 10 percent is the standard discount rate of the Development Assistance Committee (DAC) of the Organization for Economic Cooperation and Development (OECD). For more detail on the grant element of loans to the Sudan, see Yassin (1983).

Table 3

Values of the Grant Element of Official Loans and the Terms of Borrowing: A Weighted Annual Average (1958–1979)

Years	Total Amount of Contracted Loans (L.S. Million)	Average Rate of Interest (Percent)	Average Repayment Period (Years)	Average Grace Period (Years)	Average Grant Element at a Discount Rate, Based on	
					8 Percent	10 Percent
1958	13.6	5.5	17	3	25.5	35.9
1959	7.8	3	8	8	23.7	31.5
1960	9.3	5.6	15	2	21.3	31.1
1961	31.8	4.2	17	6	34.1	42.8
1962	9.7	4.2	11	4	21.8	29.6
1963	7.4	2.7	13	6	32.6	40.8
1964	.27	5.75	40	10	59.6	71.5
1965	17.025	4.8	17	4	28.8	38.6
1966	6.62	4.9	9	3	14.2	21.4
1967	27.52	3.7	12	3	25.0	31.9
1968	24.9	3.9	16	4	29.0	37.5
1969	18.0905	4.7	8	2	16.0	22.0
1970	24.225	1.9	10	10	31.3	40.5

Continued –

Table 3 – (*Continued*)

Years	Total Amount of Contracted Loans (L.S. Million)	Average Rate of Interest (Percent)	Average Repay-ment Period (Years)	Average Grace Period (Years)	Average Grant Element at a Discount Rate, Based on	
					8 Percent	10 Percent
1971	46.1	2.6	6	8	18.5	24.8
1972	64.82668	3	16	5	32.2	38.9
1973	119.332	3.2	17	7	34.5	41.7
1974	216.72	4.9	15	5	26.6	33.5
1975	116.52	3.2	22	6	40.9	48.4
1976	108.865	3	22	7	42.9	51.0
1977	85.229	3	18	5	37.6	46.6
1978	204.91	3.9	16	6	33.0	41.1
1979	42.26	2.4	32	7	54.1	62.4
Over-all Average		4	16	6	31	39

Source: Own estimates based on data from the same source as Table

Appendix 1

THE DERIVATION OF FORMULA (5) USING ZETA TRANSFORMATION FOR TRUNCATED FUNCTIONS

The zeta transformation of the discrete time-series function $f(nT)$, which was used by Beenhakker (1976) to derive formula (5), can be defined as:

$$z\left\{f(nT)\right\} = \sum_{n=o}^{\infty} f(nT) / (1+zT)^n \quad \ldots \qquad \ldots \tag{7}$$

where z is a variable, n is an integer, and T is a constant length of time. If z is replaced by the interest rate i and the constant time interval T is equal to unity (i.e., the compounding period is a unit of time), then Equation (7) can be written as

$$z\left\{f(n)\right\} = \sum_{n=o}^{\infty} f(n) / (1+i)^n \quad \ldots \qquad \ldots \tag{8}$$

which describes the present value of cash flows over time in the same manner as Equation (1). Since in loan agreements the function $f(n)$ may change over time, the notation $t = h$ will denote the time when the function starts, and $t = k - 1$ the time after which the function terminates, i.e., $f(n)$ will have non-zero values over the interval $h \leqslant n \leqslant k - 1$, and otherwise zero values. Therefore, the zeta transformation, or the present value of $f(n)$ starting at $n = h$ and ending after $n = k - 1$, can be used to determine the present value of cash loans, assuming a "constant" stream of debt-servicing payments. As such, a typical schedule of a loan repayment during a given period of time can be mapped out in the table below (where all the variables are as defined before):

Years	Interest	Debt Servicing
1	rF	–
2	rF	–
.	.	.
.	.	.
.	.	.
G	rF	–
$G+1$	–	$rF(1+r)^{N-G+1} / \left[(1+r)^{N-G+1} - (1+r)\right]$
$G+2$	–	$rF(1+r)^{N-G+1} / \left[(1+r)^{N-G+1} - (1+r)\right]$
.	.	.
.	.	.
.	.	.
N	–	$rF(1+r)^{N-G+1} / \left[(1+r)^{N-G+1} - (1+r)\right]$
N Years	rFG	$(N-G)rF(1+r)^{N-G+1} / \left[(1+r)^{N-G+1} - (1+r)\right]$

The table presents the cost of interest and a constant stream of debt-servicing payments. The total amount that the borrower has to repay during a period of N years is:

$$rFG + (N-G)\, rF\,(1+r)^{N-G+1} \;/\; \left[(1+r)^{N-G+1} - (1+r)\right]$$

It is clear that these future repayments do not reflect the true value of money at the "present" time and, therefore, these repayments must be converted into their present equivalent value and summed up. The conversion can be done by applying the zeta transformation of the function $f(n) = c$ (as shown in the above table) to:

(1) The annual amounts of rF during years 1 through G (with $c = rF$, $h = 1$ and $k - 1 = G$).

(2) The annual amounts of $rF\,(1+r)^{N-G+1} \;/\; \left[(1+r)^{N-G+1} - (1+r)\right]$ during years $G+1$ through N (with $c = rF\,(1+r)^{N-G+1} \;/\; \left[(1+r)^{N-G+1} - (1+r)\right]$, $h = G+1$ and $k - 1 = N$).

Thus, the present value related to the arrangements of this kind of repayments is:

$$PV = rF\left[1-(1+i)^{-G} + (1+r)^{N-G+1} \times \left((1+i)^{-G} - (1+i)^{-N}\right) \div \left((1+r)^{N-G+1} - (1+r)\right)\right] / i$$

Appendix 2

THE DERIVATION OF FORMULA (6)

Suppose we have two functions $f(x)$ and $g(x)$ which are zero when $x = a$. Although the ratio $f(a)/g(a)$ is an undefined quantity $[\frac{0}{0}]$, the limit of $f(x)/g(x)$ as $x \to a$ may exist nevertheless.

Consider the ratio of $f(x)$ and $g(x)$ and let both functions be expressed at the point $x = a$ by using Taylor's Theorem.

Then:

$$\frac{f(x)}{g(x)} = \frac{f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots\ldots\ldots\ldots\ldots}{g(a) + (x-a)g'(a) + \frac{(x-a)^2}{2!} g''(a) + \ldots\ldots\ldots\ldots\ldots} \qquad (9)$$

By assumption, $f(a) = g(a) = 0$, ... (10)

and therefore

$$\frac{f(x)}{g(x)} = \frac{f'(a) + \frac{(x-a)}{2!} f''(a) + \ldots\ldots\ldots\ldots\ldots}{g'(a) + \frac{(x-a)}{2!} g''(a) + \ldots\ldots\ldots\ldots\ldots} \qquad (11)$$

Hence:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} \qquad \ldots \qquad \ldots \qquad (12)$$

Provided that $g'(a)$ is non-zero, Equation (12) shows that the limit of the ratio of the two functions as $x \to a$, where both functions are equal to zero when $x = a$, is given by the ratio of the derivatives of the two functions, each being evaluated at $x = a$.

If, however, $f'(a) = g'(a) = o$, then the same procedure must be applied. Provided that the limit exists, it is usually possible to find a value of n such that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)} \qquad \ldots \qquad \ldots \qquad (13)$$

This method of evaluating limits is normally expressed by rewriting Equation (12) as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad \ldots \qquad \ldots \qquad (14)$$

which is known as "L'Hospital's Rule".

By applying this rule to formula (5) when $G = N$, the following set of equations can be obtained:

let $f(x) = (1+i)^{-x} - (1+i)^{-n}$ ... ... (15)

since $a^b = e^{b \log a}$ ... ... (16)

then: $f(x) = [e^{-x \log (1+i)} - (1+i)^{-n}]$ ... (17)

and $g(x) = (1+r)^{n-x+1} - (1+r)$ ... ... (18)

then: $g(x) = [e^{(n-x+1) \log (1+r)} - (1+r)]$ ... (19)

since $f(n) = g(n) = 0$ ... ... (20)

then: $f'(x) = -e^{-x \log (1+i)} . \log (1+i)$ ... (21)

$= -(1+i)^{-x} \log (1+i)$ ... ... (22)

and $g'(x) = -e^{(n-x+1) \log (1+r)} . \log (1+r)$ ... (23)

$= -(1+r)^{n-x+1} . \log (1+r)$ ... (24)

Therefore, $$\lim_{x \to n} \frac{f(x)}{g(x)} = \lim_{x \to n} \frac{f'(n)}{g'(n)} = \frac{(1+i)^{-n} \log (1+i)}{(1+r) \log (1+r)} \quad (25)$$

Hence, when $G = N$, the required formula would be:

$$PV = rF \left[1 - (1+i)^{-G} + (1+r) \times \left((1+i)^{-G} \log (1+i) \right) \div \left((1+r) \log (1+r) \right) \right] / i$$

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